$n \sin \beta$. From this it is evident that only one of the D and F envelopes can be cuspidal, except in a special case to be mentioned presently, when both may be so.

The Influence of β. When y_0 or $-\cos \beta \pm 1$ is small numerically, the D and F envelopes will as a rule intersect. Although unequal, they are always symmetrical on account of the positions of E and G on opposite sides of L. They are equal only when $y_0 = 0$, that is, when $\beta = 0°$ or $180°$, being coincident when n is odd, and symmetrically displaced when n is even. When y_0 or $-\cos \beta \pm 1$ is large numerically, the D and F envelopes cannot intersect, because E and G are then at very unequal distances from L and A. The inner one of the two will be cuspidal.

Illustrations. Fig. 3 with $n = 3$, $\beta = 30°$, $x_0 = n \sin \beta = 1.5$, $y_0 = -\cos \beta - 1 = -1.866$, shows the D or inner envelope cuspidal, while Fig. 4 with the same values of n, β, x_0, but with $y_0 = -\cos \beta + 1 = +0.134$, would seem at first sight to show two equal symmetrical and cuspidal envelopes. Only one however is in fact cuspidal and congruent to the inner envelope of Fig. 3. It is here the F envelope and is really smaller than the other or D envelope, because in this case $GL < EL$. The D envelope of Fig. 3 is in every way exactly equal to the F envelope of Fig. 4. When $\beta = 60°$ (but $n = 2$), as in Fig. 5, the inequality of the two envelopes is obvious, the smaller alone being cuspidal.

Transition Envelopes. Fig. 6 shows a transition envelope for $n = 3$. When the tracing point P is started in phases $0°$ with a smaller numerical value of y_0 than $-\cos \beta - 1$ as in Fig. 3, the cusp throws a lobe or a shoot like a growing bud, while the outer envelope is contracted. While y_0 is increasing in the positive direction, the two envelopes may become apparently equal or nearly so, although neither can be cuspidal. The cuspidal stage will be reached again when $y_0 = -\cos \beta + 1$ as in Fig. 4. After that the growth just mentioned will be reversed, until the value of $\beta' = 180° - \beta$ will again give cuspidal envelopes symmetrical however to those of β. For still larger positive values of y_0, the envelopes will tend to become nearly equal and circular.

Cuspidal envelope rosettes are best drawn when n is a small integer. When $n = 1$ one or both of the envelopes are cardioids for all values of β.

SEXAGESIMAL FRACTIONS AMONG THE BABYLONIANS.

BY FLORIAN CAJORI, University of California.

In view of the fact that certain writers have expressed the conviction[1] that, while the Babylonians operated with integers expressed in the sexagesimal system, they *did not use sexagesimal fractions*, it is worth while to refer briefly to a cuneiform tablet recently described by H. F. Lutz[2] of the University of Pennsylvania which unquestionably reveals the Babylonian use of sexagesimal frac-

[1] For example, see this MONTHLY, *1920*, 124.

[2] *American Journal of Semitic Languages and Literatures*, vol. 36, 1920, pp. 249–257. Tablet CBS 8536 in the University Museum, Philadelphia.

tions. According to Lutz, the tablet "cannot be placed later than the Cassite period, but it seems more probable that it goes back even to the First Dynasty period, *ca.* 2000 B.C."

To mathematicians the tablet is of interest because it reveals operations with sexagesimal fractions resembling modern operations with decimal fractions. For example, 60 is divided by 81 and the quotient expressed sexagesimally. Again, a sexagesimal number with two fractional places, 44(26)(40), is multiplied by itself, yielding a product in four fractional places, namely [32]55(18)(31)(6)(40). In this notation the [32] stands for 32 $\times$ 60 units, and to the (18), (31), (6), (40) must be assigned, respectively, the denominators 60, 60^2, 60^3, 60^4.

Numbers that are incorrect are marked by a *.

First column

. . . gal (?) -bi 40 -ám
šu *a- na* gal-bi 30 -ám

igi 2	30
igi 3	20
igi 4	15
igi 5	12
igi 6	10
igi 8	7(30)
igi 9	6(40)
igi 10	6
igi 12	5
igi 15	4
igi 16	3(45)
igi 18	3(20)
igi 20	3
igi 24	2(30)
igi 25	2(24)
igi 28*	2(13)(20)
igi 30	2
igi 35*	1(52)(30)
igi 36	1(40)
igi 40	1(30)
igi 45	1(20)
igi 48	1(15)
igi 50	1(12)
igi 54	1(6)(40)
igi 60	1
igi 64	(56)(15)
igi 72	(50)
igi 80	(45)
igi 81	(44)(26)(40)

Fifth column

	44(26)(40)
1	44(26)(40)
2	[1]28(53)(20)
3	[2]13(20)
4	[2]48(56)(40)*
5	[3]42(13)(20)
6	[4]26(40)
7	[5]11(6)(40)
9	[6]40
10	[7]24(26)(40)
11	[8]8(53)(20)
12	[8]53(20)
13	[9]27(46)(40)*
14	[10]22(13)(20)
15	[11]6(40)
16	[11]51(6)(40)
17	[12]35(33)(20)
18	[13]20
19	[14]4(26)(40)
20	[14]48(53)(20)
30	[22]13(20)
40	[29]37(46)(40)
50	[38]2(13)(20)*
44(26)(40)a-na 44(26)(40)	
[32]55(18)(31)(6)(40)	
44(26)(40) square	
igi 44(26)(40)	81
igi 81	44(26)(40)

The tablet contains twelve columns of figures. The first column gives the results of dividing 60 in succession by twenty-nine different divisors from 2 to 81. The eleven other columns contain tables of multiplication; each of the numbers 50, 48, 45, 44(26)(40), 40, 36, 30, 25, 24, 22(30), 20 is multiplied by integers up to 20, then by the numbers 30, 40, 50, and finally by itself. Using our modern numerals, we reproduce the first and the fifth columns of the tablet, which exhibit a larger number of fractions than do the other columns. The Babylonians had

no mark separating the fractional from the integral parts of a number. Hence a number like 44(26)(40) might be interpreted in different ways; among the possible meanings are $44 \times 60^2 + 26 \times 60 + 40$, $44 \times 60 + 26 + 40 \times 60^{-1}$, and $44 + 26 \times 60^{-1} + 40 \times 60^{-2}$. Which interpretation is the correct one can be judged only by the context, if at all.

The exact meaning of the first two lines in the first column is uncertain. In this column 60 is divided by each of the integers written on the left. The respective quotients are placed on the right.

In the fifth column the multiplicand is 44(26)(40) or 44 4/9. The last two lines seem to mean "$60^2 \div 44(26)(40) = 81$, $60^2 \div 81 = 44(26)(40)$."

It is a source of gratification to find that scholars of several thousand years ago were fully as capable of committing errors in computation, as are arithmeticians of the present time.

The Babylonian use of sexagesimal fractions is shown also in a clay tablet described by A. Ungnad[1] (*Orient. Lit. Zeitung,* 19 Jahrg., 1916, p. 363-368). In it the diagonal of a rectangle whose sides are 40 and 10 is computed by the approximation $40 + 2 \times 40 \times 10^2 \div 60^2$, yielding 42(13)(20), and also by the approximation $40 + 10^2 \div \{2 \times 40\}$, yielding 41(15). Translated into the decimal scale, the first answer is 42. 22 +, the second is 41.25, the true value being 41.23 +.

A BUDGET OF EXERCISES ON DETERMINANTS.

By THOMAS MUIR, Rondebosch, South Africa.

A collection of fresh exercises[2] on a mathematical subject, even if the plan and execution be far from perfect, can be of greater service to the student than any so-called "paper," covering the same amount of page space. It is in this belief that I have brought together the following budget of thirty. In a kind of way they range over the whole subject of determinants: at any rate they do not confine themselves to any special branch of it. They are of all degrees of difficulty, starting with commonplace instances of mere "evaluation." They naturally also differ in suggestiveness; one or two of them might in eager hands lead to the formulating of allied results, and thereby even to the evolution of material for a "paper." None of them, so far as I can at present recall, has been printed before, and certainly, the number of them that may so turn out must be comparatively trifling.

1. If $\begin{vmatrix} a_1 - x & a_2 - x & a_3 - x \\ b_1 - x & b_2 - x & b_3 - x \\ c_1 - x & c_2 - x & c_3 - x \end{vmatrix} = 0$, then $x = -\,|a_1b_2c_3| \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{vmatrix}$.

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[1] Our information is drawn from *Mitteilungen zur Geschichte der Medizin und der Naturwissenschaften,* Leipzig, vol. 17, 1918, p. 203.

[2] For any technical terms, in the following, which the student may have doubt about he is referred to the text-books of Weld and Hanus.